Some Trivial But Useful Mathematics

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There are some theorems in mathematics that, though quite useful, are very trivial and obvious, hence are never discussed even in elementary texbooks. But to persons with limited mathematical knowledge, who might be able to put these theorems to good use, they may be neither trivial nor obvious, and may remain unknown and unused. I would like to discuss some mathematics of this kind that relates to quantities that are scaled only ordinally, hence are defined only up to arbitrary monotonic transformations.

Ordinally scaled variables play a very modest role in the natural sciences, although there are some well known examples of their use. For instance, the Mohs scale of hardness in mineralogy orders minerals by relative hardness. The cardinal numbers assigned to various points along the scale could be replaced by any other set of numbers that preserved the ordering of the substances. There are other examples of ordinal scales in the natural sciences, but they are relatively unimportant.

The situation in the social sciences is quite different, for here ordinal scales abound. Although in economics quantities and prices of goods can be measured cardinally in a natural manner, and in demographics, sizes of populations and ages of individuals, the other social sciences often do not find it easy to discover natural cardinal units. Moreover, as we shall see, even in economics and in the natural sciences we may wish sometimes, in the face of ignorance about the precise shapes of functions, to use only ordinal properties of our variables.

Invariance under Monotonic Transformation

If a variable is meant only to define the ordering of some items along a scale, we style the variable ordinal. This variable can be replaced by any other that does not disturb the ordering by transposing items. Thus, if x ($x>0$) is an ordinal variable, we may replace it by x^2 or e^x or log x, for if $x_i>x_j$ then $x^2_i>x^2_j$, exp $x_i>$ exp x_j and log $x_i>$ log x_j.

More precisely, the ordering of a set of elements by a variable associated with that ordering is invariant under any positive monotonic transformation of the variable. Consider x ϵ X and y ϵ Y. Then f(x), X $\rightarrow$ Y, is va positive monotonic transformation if $y_i = fx_i > y_j = fx_j$ whenever $x_i > x_j$. If a set of objects has been ordered by the assignment of values of *x* then replacement of all these values by the corresponding value of *y* will not change the ordering.

For simplicity, we will confine ourselves to variables that are defined over the reals. Let *x* and *y* be a pair of such variables, and let us suppose that they are positively monotonically related. That is, there is a function f(•), f:R-->R, such that $x_i > x_j$ iff $y_i = f(x_i) > y_j = f(x_j)$. Now we subject both *x* and *y* to positive monotonic transformations, g(•) and h(•) respectively, so that z = g(x) and w = h(y). Since g and h are monotonic we have $z_i > z_j$ iff $x_i > x_j$, and $w_i > w_j$ iff $y_i > y_j$. But since *x* and *y* are positively montonoically related $x_i > x_j$ iff $w_i > w_j$. This is readily generalized to:

Theorem: If two variables *x* and *y*, are positively (negatively) monotonically related, and if *x* is transformed to *z* and *y* to *w* by arbitrary positive monotonic transformations, then *w* and *z* are positively (negatively) monotonically related.

For example, suppose we have observed empirically, using ordinal variables, that social-frustration grows with social-mobilization. Then if we replace our social-frustration scale with a new ordinally equivalent scale (social-frustration*), and our social mobilization scale with an ordinally equivalent scale (social-mobilization*), it follows that social-frustration* grows with social-mobilization*.

Moreover, monotonic relations betwen ordinal variables are transitive in the following sense: If *x* varies positively with *y* and *y* varies positively (negatively) with *z* then *x* will vary positively (negatively) with *z*, and the latter relation will be invariant under arbitrary positive monotonic transformations of *x* and *z*. The relation of equality is also obviously invariant under monotonic transformations of the variables.

Continuous, Differentiable Functions

Consider ordinal variables *x*, *y*, and *z*, and suppose we find that *x* varies positively monotonically with *y* when *z* is constant, and *x* varies negatively monotonically with *z* when *y* is constant. Then if *x* = f(*y*,*z*) is everywhere differentiable with $\partial x/\partial y > 0$ and $\partial x/\partial z < 0$.

Under the circumstances, it would be convenient if we could always simplify the function $x = f(y,z)$ by taking monotonic transformations of our variables in such a way as to transform the function to $x^* = y^*/z^*$ or $x^* = y^* - z^*$, where x^*, y^*, and z^* are the transformed variables. Is this possible in general? We can see that it is not, as follows:

Let $x = f(y,z)$ be continuous and differentiable in y and z, with $\partial x/\partial{>}0$ andd $\partial x/\partial z{<}0$ everywhere. Consider monotonic transformations of x, y, and z: $x^* = \lambda(x)$, $y^* = \varphi(y)$, $z^* = \psi(z)$, such that $x^* = y^* - z^*$. Then,

(1) $\partial x^*/\partial y^* = dx^*/dx(\partial x/\partial y)dy/dy^* = 1$

$\partial x^*/\partial z^* = dx^*/dx(\partial x/\partial z)dz/dz^* = -1$

To remind ourselves that dx^*/dx, $\partial x/\partial y$, and $\partial x/\partial z$ are functions of y and z, we write:

$dx^*/dx = \rho(y,z), \partial x/\partial y = \sigma(y,z), \partial x/\partial z = \tau(y,z)$

Since we have also defined $y^* = \varphi(y)$ and $z^* = \psi(z)$, we rewrite equation (1):

(2) $\rho(y,z)\sigma(y,z)\varphi'(y) = 1$ and $-\rho(y,z)\tau(y,z)\psi'(z) = 1$,

where $\varphi'(y) = d\varphi(y)/dy$, and $\psi'(z) = d\psi(z)/dz$.

Equating the two left-hand sides of (2), and dividing by $\rho(y,z)$, we get:

(3) $\sigma(y,z)\varphi'(y) = -\tau(y,z)\psi'(z)$, whence

(4) $-\sigma(y,z)/\tau(y,z) = \psi'(z)/\varphi'(y)$.

But equation (4) can hold only if the left-hand side, a ratio of two functions of y and z, can be factored into the ratio of a function of z to a function of y. To show this is not always possible, we exhibit the following simple counterexample: Let $x = y^2 + zy + z^2$ Then, $\partial x/\partial y = 2y + z$; $\partial x/\partial z = y + 2z$, so that $-(2y + z)/(y + 2z) = \psi'(z)/\varphi'(y)$ But there is no way to factor the left-hand side into the ratio of a function of z to a function of y. Hence $x = y^2 + zy + z^2$ cannot be changed, by monotonic transformations of the variables, into an expression of the form $x^* = y^* - z^*$.

The proof that $-(2y + z)/(y + 2z)$ cannot be factored into $\psi'(z)/\varphi'(y)$ is straightforward.[1]

[1] The idea for this proof was suggested to me by Juan Schaffer.

Set $z=0$, obtaining $\psi'(0)/\varphi'(y) = -2y/y = -2$. Now set $z=1$. Then $\psi'(1)/\varphi'(y) = (2y+1)/(y+2)$. But taking the ratio of these two expressions, φ' cancels out, leaving $\psi'(0)/\psi'(1) = 2(y+2)/(2y+1) = 1 + 3/(2y+1)$, a contradiction, since $\psi'(0)/\psi'(1)$ is a constant, independent of of *y*.

The reader may find it instructive to show that $x=y\cdot z$ and $x=y^z$ both do satisfy the condition expressed in equation (4), and that they can therefore be transformed monotonically into $x^* = y^* \cdot z^*$. The transformation obviously is obtained by taking $x^* = \log(x)$ in the first case and $x^* = \log(\log(x))$ in the second.

Now we introduce the notation:

(5) $x = [y_1 o y_2 o \ldots o y_k] / [z_1 o z_2 o \ldots o z_m]$ for

(6) $x = f(y_1,\ldots,y_k,z_1,\ldots,z_m)$ with $\partial x/\partial y_i > 0$, $i=1,\ldots,k$, and $\partial x/\partial z_j < 0$, $j=1,\ldots,m$,

these relations holding throughout some region of $R^{k+m} \rightarrow R$. Then from the transitivity of ordering under positive monotonic transformations, shown earlier, we can draw inferences of the following kind:

If $x=y/z$, $y=w/v$, $v=u/s$, $u=r/t$

Then $x = w/(z_0 v) = (w_0 s)/(z_0 u) = (w_0 s_0 t)/(z_0 r)$.

That is to say, x will vary positively with variations in w, s, and t, and negatively with variations in z and r.

An Example

The political scientist Samuel P. Huntington, in his book <u>Political Order in Changing Societies</u>, makes use of ordinal variables in a way that has excited the criticism and ridicule of some professional mathematicians although it appears to fall entirely within the framework of the analysis set forth here. On page 55 of his book he set forward the following "relationships":

(1) Social mobilization / Social frustration = Economic development

(2) Social frustration / Political participation = Mobile opportunities

(3) Political participation / Political instability = Political institutionalization

In the accompanying text, the measurement of these variables is discussed in such a way as to make it obvious that they are defined only up to monotonic transformations. Hence it certainly makes no sense, as the critics point out, to speak of ratios of these variables. But a sympathetic reading of Huntington's text reveals that, at most, he is guilty of the sin of using unorthodox notation. For in his reasoning about the social and political processes represented by these formalisms, he makes use only of the partial derivatives of the dependent on the independent variables, the derivatives being positive for variables in the numerators of his "fractions" and negative for variables in the denominator.

Huntington introduces these relations with the paragraph (p.55):

"In these conditions, political participation becomes the road for advancement of the socially mobilized individual. Social frustration leads to demands on the government and the expansion of political participation to enforce those demands."

[Translate: social mobilization-->social frustration-->political participation.]

"The absence of mobility opportunities and the low level of political institutionalization in most modern countries produce a correlation between social frustration and political instability."

[Translate: without high compensating levels of mobility and adaptability of political institutions, political participation-->political instability.]

Thus, Huntington's formalism, however we might wish to modify his notation, provides a concise summary of the relations among the partial derivatives of seven variables decribing three social mechanisms. To support the analysis, only ordinal measurement of the variables is required.

Dynamic Systems with Ordinal Variables

Thus far we have been concerned with static functional relations among variables. Now we turn to time-dependent relations expressable as differential equations. The discussion will be limited to the simplest case of first-order non-linear differential equations in three variables, including time. Any such system can be written as:

(7) $dx/dt = f(x,y)$

(8) $dy/dt = g(x,y)$

Taking the ratio of (8) to (7), we also have:

(9) $dy/dx = g(x,y)/f(x,y)$

Now in the x-y plane, we can draw the direction field that describes the paths of the system from any initial conditions. At any point in such a path, the slope of the path will be given by equation (9). Figure 1 depicts the direction field for (9) in the case where $g(x,y) = -y + a \log(x)$, and $f(x,y) = -x + b \log(y)$.

If we set $dx/dt = 0$, then we obtain the curve, $f(x,y) = 0$, along which the paths of direction-field are horizontal. Similarly, if we set $y = 0$, we obtain the curve, $g(x,y) = 0$, along which the paths of these direction fields are vertical. The points where $dx/dt = dy/dt = 0$ are the (stable and unstable) equilibrium points of the sytem. In Figure 1, if the paths for increasing *t* converge on an equilibrium point, the equilibrium will be stable; if they diverge from one; it will be unstable. At saddle points, some paths may converge while others diverge, making the equilibrium unstable.

The direction paths may also form closed curves in the x-y plane, so that the sytem describes a periodic motion around the curve. Such paths are called limit cycles. They are stable if neighboring paths all converge to them (from both inside the closed curve and outside), unstable otherwise.

It is obvious that these properties -- the distribution of equilibrium points and limit cycles, of a system of differential equations, and the stability or instability of these equilibria and cycles -- are invariant under ordinal transformations of *x* and *y*. For consider such a transformation, $x^* = \varphi(x)$, $y^* = \psi(Y)$. Since the orderings of x-coordinates and of y-coordinates are not altered, all direction paths will remain intact. If they converged on a point or limit cycle in the x-y plane, they will converge in the x^*-y^* plane.

Consider the case where the system defined by equations (7) and (8) has an equilibrium point which, by transformations of the coordinates and without loss of generality, we can place at the origin. If we expand $f(\bullet)$ and $g(\bullet)$ in Taylor's series and neglect the higher-order terms, the sytem is approximated by

(10) $dx/dt = ax + by$

(11) $dy/dt = cx + dy$

Integrating these equations, we obtain:

$$x = A\exp(\lambda_1 t) + B\exp(\lambda_2 t)$$

$$y = C\exp(\lambda_1 t) + D\exp(\lambda_2 t)$$

where $\lambda = \{(a+d) \pm [(a+d)^2 - 4(ad-bc)]^{1/2}$

The equilibrium will be stable iff both solutions for λ have negative real parts. But this condition will hold, in turn, iff $(a+d)<0$ and $(bc-ad)>0$. From (10) $(dy/dx)_{dx/dt=0} = -a/b$, while $(dy/dx)_{dy/dt=0} = -c/d$. So stability depends on which of these two slopes is the greater. If, for example, a and d are negative, while b and c are positive, stability requires that $bc<ad$, whence $-c/d < -a/b$. Then the slope of the curve for $dx/dt=0$ must be steeper than the slope of the curve for $dy/dt=0$.

For, by the chain rule for differentiation,

$$(dy^*/dx^*)_{dx/dt=0} = dy^*/dy(dy/dx)_{dx/dt=0}(dx/dx^*),$$

and

$$(dy^*/dx^*)_{dy/dt=0} = dy^*/dy(dy/dx)_{dy/dt=0}(dx/dx^*)$$

Subtracting these two quantities, we have:

$$[(dy^*/dx^*)_{dx/dt=0} - (dy^*/dx^*)_{dy/dt=0}] = (dy^*/dy)(dx/dx^*)[(dy/dx)_{dx/dt=0} - (dy/dx)_{dy/dt=0}].$$

Since the first two factors on the right side are always positive, the sign of the difference of the slopes of the transformed variables is the same as the sign of the difference of the slopes of the original variables. Hence, stability is preserved under monotonic transformations of the variables.

Extreme Values

Let $y=f(x)$ be a single-valued function of x, and $x^* = \varphi(x)$ and $y^* = \psi(y)$ be positive monotonic transformations of *x* and *y*, respectively. Let f(x) have a local maximum at x_0, so that for *x* in some interval about x_0, $y(x_0) = y_0 > y(x)$. Now $y_i^* > y_j^*$ iff $y_i > y(x)$. Hence, $y^*(x_0^*) > y^*(x^*)$, where $x_0^* = \varphi(x_0)$ and $x^* = \varphi(x)$ for any other x in the interval. It follows that maxima of the original function correspond to maxima of the transformed function.

This result may seem counterintuitive, since whether a stationary value of a differentiable function is a maximum or a minimum depends on the sign of the second derivative, and this sign is not, in general, invariant under prositive monotonic transformations of the variables (concavities can change to convexities, and vice versa).

However, it is easy to show that the sign of the second derivative *is* invariant in the neighborhood of a stationary point. The proof again makes use of the chain rule for differentiation. Assume the essential conditions of continuity and twice differentiability.

Suppose that for $x = x_0$, $dy/dx = 0$, $d^2y/dx^2 < 0$. Define $x^* = \varphi(x)$, $y^* = \psi(y)$.

(12) $dy^*/dx^* = (dy^*/dy)(dy/dx)(dx/dx^*)$, hence,

(13) $(dy^*/dx^*)_{dx/dt=0} = 0$

(14) $d^2y/dx^2 = d/dx\{dy/dx\}$

(15)
$$\begin{aligned} d^2y^*/dx^{*2} &= d/dx^*\{dy^*/dx^*\} \\ &= (dx/dx^*)d/dx\{dy^*/dx^*\} \\ &= (dx/dx^*)d/dx\{dy^*/dy(dy/dx)(dx/dx^*)\} \\ &= (dx/dx^*)[d/dx\{dy^*/dy\}(dy/dx)(dx/dx^*] \\ &\quad + d^2y/dx^2(dy^*/dy)(dx/dx^*) \\ &\quad + d/dx\{dx/dx^*\}(dy^*/dy)(dy/dx)] \end{aligned}$$

But, since for $x = x_0$, $dy/dx = 0$, this reduces for $x = x_0$, $x^* = x_0^*$, to:

(16) $(d^2y^*/dx^{*2})_{x^*=x_0^*} = (dx/dx^*)(d^2y/dx^2)_{x=x_0}(dy^*/dy)(dx/dx^*)$

Since $dx/dx^* > 0$, $dy^*/dy > 0$, and $(d^2y/dx^2)_{x=x_0} < 0$,

we have $(d^2x^*/dx^{*2})_{x^*=x_0^*} < 0$.

Comparative Statics

Much economic analysis takes the form of comparative statics. The (stable) equilibrium of a system is displaced by a change in one of its parameters, and we wish to know how the values of the system variables are changed when it settles to its new equilibrium. Moreover, much comparative statics analysis in economics is qualitative, in the sense that the shapes of the system functions may not be known, but only the signs of their derivatives, and in the sense that only the sign of the disturbance, and not its magnitude, may be given.

To see what kinds of conclusions can be reached under such circumstances, let us consider a simple example. For a certain commodity, the quantity, q, will be supplied by producers if the price is $p_S = p_S(q)$, and the same quantity will be purchased by consumers if the price is $p_D = p_D(q)$. More will be supplied if the price is higher ($dp_S/dq>0$), and less will be purchased if the price is higher ($dp_D/dq<0$). The market will be in equilibrium when $p_S(q_0) = p_D(q_0) = p_0$.

Now suppose that a sales tax is imposed on the commodity, so that the net supply price for any quantity is increased by the amount of the tax. Call the new supply price $\tilde{p}_S = \tilde{p}_S(q)>p_S(q)$, and $\tilde{p}_S(\tilde{q}_0) = p_D(\tilde{q}_0) = \tilde{p}_0$.

We can now show that the equilibrium price will be increased, $\tilde{p}_0>p_0$, and the equilibrium quantity will be decreased, $\tilde{q}_0<q_0$. Moreover, since the tax is such that $\tilde{p}(q)$ increases with an increase in q, the equilibrium increase in price will be smaller than it would have been if the quantity exchanged had remained constant:

$\tilde{p}(q_0)>\tilde{p}(\tilde{q}_0) = \tilde{p}_0$.

These results do not depend on continuity or differentiability of the functions. For all $q>q_0$, $p_D(q)<p_0$, since the demand price decreases when the quantity increases. But $p_S(q)>p_0$, since the supply price increases under the same conditions.

For all q, $\tilde{p}_S(q)>p_S(q)$, the difference being the amount of the tax. Therefore, $p_D(q)<p_0<p_S(q)<\tilde{p}_S(q)$ for all $q>q_0$, and $p_D(q) = \tilde{p}_S(q)$ in this range of q. Hence, if a new equilibrium exists, we must have $\tilde{q}_0<q_0$. Now for all $q<q_0$ (including $\tilde{q}_0$), $\tilde{p}_S(q)<\tilde{p}_S(q_0$, hence $\tilde{p}_S(q)>\tilde{p}_S(q_0) = \tilde{p}_0$. Similarly, $p_D(q)>p_D(q_0$, so that $p_D(\tilde{q}) = \tilde{p}_0>p_0$.

Conclusion

In these pages I have sketched out some of the properties of continuous, differentiable functions that invariant under monotonic transformations of the variables.

In the case where two variables are positively (negatively) monotonically related, their positive monotonic transforms are positively (negatively) monotonically related.

If x is a monotonic function of y and y of x, then x is a monotonic function of z. If we assign the number zero to a monotonic relation if it is positive, and 1 if it is negative, and if x_1, $x_1, x_2, x_3, \ldots x_n$ are a sequence of variables, each of which is a monotonic function of its successor, than the sign of the relation between x_1 and X_n will be positive or negative as the sum, modulo 2, of the numbers assigned to the intervening relations is 0 or 1.