Residential Choice and Air Pollution: A General Equilibrium Model

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Pigou's parable of the smoke belching factory imposing an externality on the neighboring laundry has elicited more controversy than one could have expected from such a simple situation. Ronald Coase claimed that the Pigouvian solution of taxes and subsidies was demonstrably inefficient while William Baumol recently defended Pigou by presenting a situation in which a tax placed upon the factory (without taxation or compensation to the laundry) optimizes resource allocation under pure competition.

What if the externality affects the factory itself? Suppose the smoke reduces the work efficiency of employees and causes ill health. It might be conjectured that nothing new is gained by adding the wrinkle of poisonous air which sickens factory workers, since the factory both generates the smoke and is affected by it. By analogy, a "smoke city" where everyone works in factories generating air pollution (which affects only the city) might be deemed consistent with Pareto optimality, presuming the factories compensate individuals for the air pollution with higher wages.1 Similarly, recent policy discussions have advocated "exporting" air pollution by importing pollution-producing goods. Little concern has been given to the welfare implications since the "republic of smoke" would be experiencing the profit as well as the pollution from production. There are even proponents of the view that some regions of the United States ought to be kept quite clean, while others are allowed to become highly polluted. Their reasoning is that if workers are mobile and acting in a competitive market, they will demand and receive higher wages for living in the polluted locality and the resulting situation will be Pareto optimal.

This paper will show these arguments to be fallacious. For externalities such as air pollution, it is precisely the assumptions of competition and mobility which destroy Pareto optimality. Restoration of efficiency can occur only through outside intervention designed to alter factor prices.

I. The Model

Suppose there exists an economy comprised of two cities: one producing a good which does not cause air pollution (we shall name it bread and refer to the area as bread city); the other manufacturing a commodity which produces air pollution as a by-product (we shall name it steel and refer to the region as steel city). Further, we assume a population of individuals (normalized so that the total population is equal to 1)3 allocated between the two cities with θ living and working in steel city and 1−θ inhabiting bread city.4 The problem is determining whether with free movement between the cities (no transportation or moving costs), the wage rates will adjust so that the total population will allocate itself in a Pareto optimal fashion.5

The basic framework of this model is similar to the transportation parables of Robert Strotz.

Note, in checking the units of the various expressions we will derive, this normalization results in an occasional implicit multiplicative factor of 1 which has the units of population.

An alternative interpretation is to regard the total population as allocating θ percent of their time to living and working in steel city and 1−θ percent of their time to living and working in bread city.

In our model the particular Pareto optimum will correspond to the maximum of a linear social welfare function, see below.

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1 This assumes competition and mobility among workers.

2 The Urban Institute. I am deeply indebted to Nancy M. Gordon, Lester B. Lave, Robert E. Lucas, Jr., and Thomas E. Morton for helpful comments and suggestions. I am especially indebted to Herbert A. Simon, who is co-author of the Appendix section.

3 In our model the particular Pareto optimum will correspond to the maximum of a linear social welfare function, see below.
Using (A5) and (A7), we can rewrite (A6) as:

\[
\frac{du^i}{u^i_2} = dp^i - s dp^i + \left( \frac{u^i_2}{u^i_1} \right) da^i
\]

or, more explicitly, for \( B \) and \( S \) separately:

\[
\frac{du^i}{u^i_2} = -s dp^i \quad (i \in B)
\]

\[
\frac{du^i}{u^i_2} = (1 - s)dp^i + \left( \frac{u^i_1}{u^i_2} \right) d\theta \quad (i \in S)
\]

Since \( dp^i > 0 \) and \( s^i > 0 \), we see from (A9) that the population shift has lowered the utility of all residents of bread city. Since \( s^i < 1, u^i_1 < 0, u^i_2 > 0 \) and \( d\theta < 0 \), we see from (A10) that the population shift has increased the utility of all residents of steel city. Moreover, since the dimensionality of the terms in (A9) and (A10) is "loaves," we can determine immediately how large a subsidy or assessment, in the same units, would restore utilities to their previous levels.

We now give a subsidy to or impose an assessment on each person of such magnitude that his utility will be at least equal to its initial level. If this scheme is to be feasible then the sum of the assessments must be equal to or greater than the sum of the subsidies (the government cannot redistribute more than it collects).

We wish to show that:

\[
\sum_U e^i = \sum_B e^i + \sum_S e^i < 0
\]

\[\text{Note, the surplus is exactly equal to the market value (i.e., bread equivalent) of the benefits obtained by residents of steel city from the lower pollution resulting from the reduction in population and steel production. The higher costs of steel to residents of bread city can be exactly compensated by assessing the higher cash incomes of residents of steel city.}\]
A. The Utility Function

Each individual in the population is assumed to have an identical utility function. Since conditions in the two cities differ (air pollution is not present in bread city), we shall represent the utility for a resident of steel city as shown in equation (1):

\[ u^* = u(s^*, b^*, a) \]

where \( s^* \) is the steel consumption per capita in steel city, \( b^* \) is the per capita bread consumption in steel city, and \( a \) is the amount of air pollution to which an individual in steel city is subjected. Similarly, utility for each bread city resident is:

\[ u^b = u(s^b, b^b, 0) \]

The amount of bread and steel consumed by the individual enters each utility function positively, \( u_s^* \) and \( u_b^* \) (where the subscripts refer to the partial derivatives with respect to that argument). The partial derivative with respect to \( a \), \( u_a^* \), is negative, indicating that air pollution is a "bad."

B. Production Possibilities

The only resource available for production of bread and steel is labor. Thus, there is a fixed aggregate quantity, 1 (total population), of this homogeneous productive resource available to the economy. The distribution of population will uniquely determine the production of bread and steel. Both commodities are produced in perfectly competitive industries under constant returns with each individual producing one unit of the relevant commodity, i.e., each bread worker produces one unit of bread. Total bread output is equal to the population of bread city, \( 1 - \theta \), and is distributed such that \( \theta b^b \) represents the population of steel city multiplied by the bread consumption per capita for bread city residents and \((1 - \theta)b^b \) is the population of steel city multiplied by the bread consumption per capita for steel city residents. An analogous interpretation applies to steel production. Thus, the production constraints in per capita terms are:

\[ \theta b^b + (1 - \theta)b^b = 1 - \theta \quad \text{(bread)} \]
\[ \theta s^* + (1 - \theta)s^b = \theta \quad \text{(steel)} \]

For simplicity we assume that each steel worker not only produces a unit of steel, but also produces a unit of pollution in the making of steel (\( a = \theta \)); henceforth \( \theta \) will replace \( a \) as the value of the third argument in the utility function for the people of steel city. Thus, the number of units of pollution is identical to the number of units of steel, which in turn is equal to the population of steel city.

II. Individual Utility Maximization

Let each individual be paid the value of his marginal product, i.e., the steel worker's income is \( 1x\theta p^* \) (one unit multiplied by the price of steel) and the bread worker's income is \( 1x\theta p^b \) (one unit multiplied by the price of bread). The consumer is then faced with maximizing his utility subject to his budget constraint. We formulate the respective Lagrangeans:

\[ u(s^*, b^*, \theta) + K^*(p^* - p^s^* - \theta p^b) \quad \text{(steel city)} \]
\[ u(s^b, b^b, 0) + K^b(p^b - \theta s^b - p^b b^b) \quad \text{(bread city)} \]

where \( K^* \) and \( K^b \) are Lagrange multipliers. We then take the partial derivatives with respect to steel and bread consumption, the variables at the consumer's disposal:

\[ u_{s^*} = K^* p^* \]
\[ u_{b^b} = K^b p^b \]

Note, the equalities in equations (3) and (4) imply full utilization of the goods.

One can think of individuals owning shares in the bread and steel factories.
Dividing (7) by (8) and (9) by (10) we obtain:

\[
\frac{u^*_s - K^*_p}{u^*_b - K^*_p} = \frac{\theta^*_s}{\theta^*_b} = 1 - \frac{b^*}{s^*}
\]

This, of course, is the familiar fact that the ratio of the marginal utilities from steel and bread must equal the ratio of the respective prices at the maximum. Furthermore, we require that \( \theta^* \) be such that the utilities in the two cities are equal to each other, i.e., \( u(s^*, b^*, \theta^*) = u(s^b, b^b, 0) \). If this were not the case, people would move from one city to another. This would violate our notion of equilibrium since total welfare could be increased by such migration. We denote the solution values of our equilibrium conditions by \( s^8, b^8, \theta^8, s^b, b^b, \) and \( \theta^b \).

**III. Conditions for a Pareto Optimum**

Pareto optimality will be achieved if each consumer’s utility is a maximum given the utility levels of all other consumers. Following the approach set forth by Strotz, we shall alternatively interpret the problem so as to maximize the linear social welfare function,

\[
\sum_s u(s^*, b^*, \theta) + \sum_b u(s^b, b^b, \theta)
\]

\[= \theta u(s^*, b^*, \theta) + (1 - \theta) u(s^b, b^b, 0)\]

given the available resources (total labor supply) and the production alternatives open to society (equations (3) and (4)). The equality comes from the fact that all individuals are identical, i.e., the welfare weights are assumed to be unity. Mathematically we set up the Lagrangean:

\[
\begin{align*}
\frac{\partial}{\partial s^*} [\theta u(s^*, b^*, \theta) + (1 - \theta) u(s^b, b^b, 0)] + \mu [\theta^* + (1 - \theta)(s^b - \theta)] + \gamma [\theta^* + (1 - \theta) s^* - \theta] \\
+ \delta [u(s^*, b^*, \theta) - u(s^b, b^b, 0)]
\end{align*}
\]

where \( \mu, \gamma, \) and \( \delta \) are Lagrange multipliers insuring that the constraints are not violated. Note the last constraint is derived from our requirement that the utilities in the two cities be equal to each other.

Differentiating (12) with respect to \( s^*, b^*, s^b, b^b, \) and \( \theta \), respectively:

\[
\begin{align*}
\frac{\partial}{\partial s^*} u^* &= \frac{\partial}{\partial s^*} u^b + \gamma \theta = 0 \\
\frac{\partial}{\partial b^*} u^* &= \frac{\partial}{\partial b^*} u^b + \mu \theta = 0 \\
\frac{\partial}{\partial \theta} u^* &= \frac{\partial}{\partial \theta} u^b + \gamma (1 - \theta) = 0 \\
\frac{\partial}{\partial \theta^*} u^b &= \frac{\partial}{\partial \theta^*} u^b + \mu (1 - \theta) = 0 \\
u(s^*, b^*, \theta) &= u(s^b, b^b, 0) + \theta u^s + \mu (b^* - b^b) + 1 \]
\]

Dividing (13) by (14) and (15) by (16) we obtain:

\[
\frac{u^*}{u^b} = \frac{\gamma}{\mu}
\]

Rearranging (17), using (14), the production constraints (equations (3) and (4)), and the requirement that the utilities in the two cities are equal, we find:

\[
\frac{\gamma}{\mu} = \frac{1 - b^b}{s^b} = \frac{u^* s^b}{u^b s^b}
\]

Equating (18) and (19) we derive a necessary condition for an optimum:

11 The last equality is derived from the income constraint for bread city residents in equation (6). The partial derivatives of the steel city and bread city utility functions are evaluated at \( (s^*, b^*, \theta) \) and \( (s^b, b^b) \), respectively.

12 In this expression, \( s^* \) has the units of steel, not steel per person. See fn. 3.

13 This is the price ratio which will induce "optimal" behavior from the consumers in the two cities. A hat over a variable denotes the value attained at the optimum. In this case, the derivatives of the steel-city and bread-city utility functions are evaluated at \( (\hat{s}^*, \hat{b}^*, \hat{\theta}) \) and \( (\hat{s}^b, \hat{b}^b, 0) \), respectively.
IV. Individual Maximization and Pareto Optimality

We now ask whether individual utility maximization, subject to fixed unit prices common to all individuals, will lead to a Pareto optimum.\(^{14}\) To do this it is sufficient to compare \(\bar{\theta}\) with \(\hat{\theta}\). If they are not equal, production will be incorrect, i.e., too much or too little steel will be produced. More precisely, we shall prove that \(\bar{\theta} > \hat{\theta}\), i.e., optimality requires a redistribution of population from steel city to bread city with less steel production (and less pollution).

First, we examine the expenditures of a bread city resident in the two situations in terms of bread units:

\[ E_b = b^* + \frac{p^*}{\hat{p}} \]

where \(E_b\) represents the number of units of bread a resident could buy if no steel was bought.

Substituting the price ratio corresponding to individual maximization (using equation (11)) into equation (21) we obtain:

\[ E_b = 1 \]

Hence, the income of bread workers (in this case, the price of bread) is equal to unity.

Similarly, we use the price ratio at an optimum (from equation (20)) and substitute it into equation (21) and find:

\[ E_b = 1 - \frac{a_1^* b^*}{a_2^*} \]

Remembering that \(a_2^* < 0\), we see optimality requires that the income in units of bread for bread workers exceed unity.\(^{15}\)

\(^{14}\) Note that the Pareto situation is a generalization of the case of individual utility maximization with the removal of the constraint that expenditures equal wage income. Therefore, by the Le Chatelier principle, the Pareto situation must be at least as good as individual maximization; if it differs it must be superior.

\(^{15}\) Mathematically, if \(\frac{p^*}{\bar{p}} \leq \frac{p^*}{\hat{p}}\) then \(\bar{\theta} > \hat{\theta}\) in order that \(a^* \geq a^*\).
equations (11) and (20) this means we will be examining cases for which the following holds:

\[
\frac{\dot{u}_1^x}{u_1^x} = \frac{\dot{u}_2^b}{u_2^b} < \frac{\dot{u}_1^b}{u_1^b} = \frac{\dot{u}_2^b}{u_2^b}
\]  

(24)

Using (24), the production constraints (equations (3) and (4)), the fact that \(d' = d'' = u_b\), the assumption of diminishing marginal utility, and the neutrality of pollution, we can enumerate the admissible changes in our variables:

| \(d\) | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + |
| \(b\) | 0 | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + |

We see the only possible changes involve a reduction in \(d\). Without some externally imposed adjustment, i.e., tax, the results from individual utility maximization will not adequately reflect the externality associated with steel production. Individuals moving to steel city will not take account of the extra pollution they subject all steel city residents to and the externality will not be completely internalized. Thus, there will be a misallocation of resources (labor) between the two cities.

18 By neutrality, we mean that a change in the pollution level does not affect the relative marginal utility of bread and steel consumption.

19 The table indicates the direction of change of the variable from individual maximization to optimality, e.g., \(r\) followed by a \(+\) means that \(r^\ast > r\). For example, looking at the first column, we see that both \(d\) and \(b\) may increase so as to satisfy the relation in (24) pertaining to bread city, i.e., that the ratio of marginal utilities increases. At the same time, it is clear that this will mean that \(d^\ast \geq d^\ast\) since more of both goods are being consumed. A consistent solution for steel city would involve a reduction in \(r\) with \(b\) remaining constant. This would necessitate a decrease in \(\theta\). \(\theta\) could not increase or remain unchanged if both \(r\) and \(b\) rise and our relationships are satisfied. Similar arguments can be made for the other columns in the table.

V. The Tax

Comparing equations (22) and (23), we see that an income transfer from residents of steel city to residents of bread city must take place to attain optimality from individual maximization. The magnitude of this transfer (in terms of bread units) is given by:

\[
\tau = \frac{E_b^b - E_b^b}{\theta} = -\frac{a_b^* \theta}{a_b^*} - \frac{\dot{a}_b^*}{\dot{a}_b^*} \theta > 0
\]  

(25)

This quantity can be viewed as a tax on and subsidy to the factors of production. Thus, steel producers are taxed for employing the "pollution-creating" input (steel labor), while bread producers are subsidized for utilizing the "pollution-free" input (bread labor). The resulting new demand curves for labor lead to an optimal reallocation of resources.

In our case, there is another interpretation to the assessment. The numerator of the tax collected from everyone in steel city (\(\theta\) people) represents the marginal disutility per person (for all people) of a change in the pollution level and the denominator represents the marginal utility of bread per person in steel city. Hence, the ratio measures the disutility of pollution relative to the utility of bread for people in steel city. When multiplied by the population of steel city, this reflects the total amount of bread they should be willing to give up to decrease the level of pollution.

V. A Numerical Example

This conclusion can be made clearer through the use of a numerical example. We shall derive the results for individual utility maximization and for a Pareto optimum utilizing the following Cobb-Douglas utility functions:

\[
u^* = \theta(\alpha^*)^{1-\delta}(\beta^*)^\delta
\]  

(26) \((\text{steel city})\)

\[
u^b = (\alpha^b)^{1-\delta}(\beta^b)^\delta
\]  

(27) \((\text{bread city})\)

20 Alternatively, it can be looked upon as an inducement to the labor force to change their working patterns. See fn. 4.

21 Note, for bread city, \(\theta=0\) and \(\alpha^b=1\) which results in (27).
We can now use these functions, together with the production constraints (equations (3) and (4)), the budget constraints (equations (5) and (6)), and the condition in equation (11) to solve a system of simultaneous equations for the unknowns $s^a$, $s^b$, $b^a$, $b^b$, and $P$ in terms of $\beta$ and $\theta$:

\begin{align*}
(28) \quad & s^a = 1 - \beta \\
(29) \quad & s^b = \frac{\beta \bar{\theta}}{1 - \bar{\theta}} \\
(30) \quad & b^a = \frac{(1 - \bar{\theta})(1 - \beta)}{\bar{\theta}} \\
(31) \quad & b^b = \beta \\
(32) \quad & P = \frac{(1 - \beta)(1 - \bar{\theta})}{\beta \bar{\theta}}
\end{align*}

Substituting these values into (26) and (27) and remembering that in equilibrium the utilities in the two cities must be equal we obtain:

\begin{align*}
(33) \quad & e^{-\gamma} \left( \frac{1 - \bar{\theta}}{\bar{\theta}} \right) = \frac{\beta}{1 - \beta}
\end{align*}

Letting $\beta = .38$, we can solve (28) through (33) for approximate numerical values of the variables. We find that steel consumption per capita in steel city ($s^a$) is .62, steel consumption per capita in bread city ($s^b$) is .38, bread consumption per capita in steel city ($b^a$) is .62, and bread consumption per capita in bread city ($b^b$) is .38. The relative price ratio ($P$) is 1.63. The total population is equally distributed between the two cities ($\theta = .50$). Finally, the level of utility for each individual in society ($u^a = u^b$) is .38.

Now, we solve for the optimum values using the Cobb-Douglas utility functions and relation (20):

\begin{align*}
(34) \quad & s^a = \frac{1 - \beta}{1 + \theta - \bar{\theta}}
\end{align*}

For $\beta = .38$, we solve (34) through (39) for the numerical values of the unknowns. In this case, steel consumption per capita in steel city ($s^a$) is .50, steel consumption per capita in bread city ($s^b$) is .33, bread consumption per capita in steel city ($b^a$) is .75, and bread consumption per capita in bread city ($b^b$) is .50. The relative price ratio ($P$) is 2.45. Less of the total population now resides in steel city ($\theta = .40$); hence, there is less steel production and less pollution. The level of utility for each individual ($u^a = u^b$) has now increased to .39. The tax ($T$) imposed on steel city residents and transferred to each bread city inhabitant is .32.

VII. Discussion

We have posed a simple market situation in which an externality (air pollution) arises as an explicit joint product in the manufacture of a "normal" good. People engaged in the production of the good are subjected to the externality, but are free to move to another pollution-free environment and produce a different good. Our model has dealt with the polar case of completely separate production locations. In this way, we allowed the market maximum freedom in coping with the problem; it was unable to do so in an optimal manner. Permitting bread production to take place in steel city would further aggravate the situation.

Since we are only concerned with relative prices, we define $P = p^a/p^b$. Note, too, that initially we are specifying a given distribution of population $\theta$ (and, hence, income). We will then search over $\bar{\theta}$ so as to equate the individual utilities in the two cities.

This, of course, is also the level of pollution in steel city.

$^{24}$ Our model has dealt with the polar case of completely separate production locations. In this way, we allowed the market maximum freedom in coping with the problem; it was unable to do so in an optimal manner. Permitting bread production to take place in steel city would further aggravate the situation.
the market fails to allocate optimally the resources (population) without outside intervention. In particular, a tax on steel city residents or a subsidy to bread city residents is needed. If this activity is to be neutral with respect to distributional effects, there must be both a tax on steel city residents and a subsidy to bread city residents.

The essential element in the model which causes the market to fail is the fact that the externality takes the form of a pure public good (or, more precisely, a "pure public bad"). Allowing some people the choice of consuming more goods and subjecting themselves to more pollution does not lead to a Pareto optimum. The final person moving to steel city does not take account of the additional pollution he inflicts upon all steel city residents.

This result is of interest when one considers suggestions that we allow some cities to become the sewers for American production or that we export our pollution by importing pollution-producing goods. Clearly, these solutions are nonoptimal for society as a whole. Apparently, sewer city would be too polluted for Pareto optimality.

APPENDIX 25

The nonoptimality theorem was proved under the rather strong assumptions that all individuals possessed identical cardinal utility functions. For it will be recalled that the original Pareto optimality conditions were derived by summing the individual utility functions. Here we allow each individual to possess his own differentiable utility function defined only up to a smooth monotonic transformation. Again, we assume positive marginal utilities for bread and steel, and negative marginal utilities for pollution. Moreover, we will use only the signs, not the marginal utilities in our derivation, and these signs are invariant under monotonic transformations of the functions. Hence, our results do not depend upon the cardinal scales of the utility functions, and we can regard these functions as defined ordinally.

Let each individual, , have a differentiable utility function:

\[ u^i = u'^i(s^i, b^i, a^i) \]

Again,

\[ a^i = 0 \] for residents of bread city \((i \in B)\)
\[ a^i = 0 \] for residents of steel city \((i \in S)\)

Each person is constrained to spend no more than his income, which is equal to the value of his product plus (minus) his subsidy (assessment):26

\[ p^i - p^i s^i - b^i + e^i = 0 \]

where \( p^i = 1 \) for \( i \in B \); \( p^i = p^r \) for \( i \in S \); and \( e^i \) is the subsidy (assessment) of the \( i \)th individual.27

Necessary conditions for competitive equilibrium again imply:

\[ \frac{u_1^i}{u_2^i} = p^r \] for all \( i \)

Now assume that some individuals are moved from steel city to bread city, so that there is a decrease, \( d\theta \), in the population of the former city and in the production of steel, and a corresponding increase in the population of the latter city and in the production of bread. As a result of the decrease in steel production there is an increase, \( dp^r \), in the relative price of steel.28 We now calculate the consequent changes in utility:

\[ du^i = u'^i ds^i + u_i db^i + u_i da^i \]

where \( da^i = 0 \) for \( i \in B \), and \( da^i = d\theta \) for \( i \in S \).

From (A4), the derivative of the budget constraint with no subsidy or assessment is:

\[ dp^i - p^i ds^i - s^i dp^r - db^i = 0 \]

26 We assume that the government may assess (subsidize) any member of the population, and that the assessments (subsidies) may be different for different persons.

27 Note, for simplification, we take the price of bread as numeraire, while the price of steel (per unit of bread) is \( p^r \).

28 We assume that a decrease in the production of steel and an increase in the production of bread always causes an increase in the price of steel relative to the price of bread. This condition is stated independently of the utility functions.